

The University of Chicago

RULED SURFACES WITH INTERSECTING GENERATORS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1939

By

RALPH NATHANAEL JOHANSON

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RULED SURFACES WITH INTERSECTING GENERATORS

Introduction. The configuration composed of two ruled surfaces whose generators are in one-to-one correspondence has been studied by Lane¹ for the case in which corresponding generators are not coplanar. He has shown that it is possible to base a projective theory of this configuration on a system of four ordinary linear first-order differential equations in four dependent variables. The purpose of this paper is to study the case in which corresponding generators intersect. It is shown that a projective theory for this configuration can be based on an analogous system of differential equations.

In Section 1 a system of four fundamental differential equations is derived and the effect on this system of the most general transformation preserving the configuration is studied.

Section 2 is devoted to a consideration of pairs of ruled surfaces for which the tangent line of the curve of intersection is not coplanar with the generators at each point. These surfaces are referred to as non-special pairs of ruled surfaces. By choosing the fundamental curves of reference suitably, it is possible to reduce the fundamental system of differential equations to a relatively simple canonical form. The most general transformation which preserves this canonical form is obtained. The

¹E. P. Lane, Ruled surfaces with generators in one-to-one correspondence, Transactions of the American Mathematical Society, vol. 25 (1923), pp. 281-296.

following section takes up the case of pairs of ruled surfaces for which the tangent line to the curve of intersection is coplanar with the generators at each point. These surfaces are called special pairs of ruled surfaces. A canonical form for the differential equations is obtained and the most general transformation preserving this form is considered.

Section 4 is devoted to the study of ruled surfaces with intersecting generators associated with a proper non-ruled surface in ordinary space. Examples of non-special and special pairs of ruled surfaces are discussed.

In the final section the Klein-Segre correspondence between lines in ordinary space and points on a hyperquadric in space S_5 is applied to study ruled surfaces with intersecting generators. According as a certain ruled surface on Q_4 is or is not developable, it is possible to distinguish special and non-special pairs of ruled surfaces in S_3 . A characteristic property is given by means of which special pairs of ruled surfaces may be considered as limiting cases of non-special pairs of ruled surfaces.

1. The differential equations. In a linear space S_3 let the four homogeneous coordinates $y^{(1)}, \dots, y^{(4)}$ of an arbitrary point P_y on a curve C_y be given as single-valued analytic functions of one independent variable t . Similarly, let the four coordinates of each of three other points P_z, P_p, P_σ be given as functions of the same variable t . Then for each value of t let us join P_y and P_p by a straight line λ_{yp} ; and let us also join P_z to the same P_p by a line λ_{zp} . As t varies, the locus of λ_{yp} is a ruled surface R_{yp} , and the locus of λ_{zp} is a ruled surface R_{zp} . Consequently, associated with each value of t there are two generators, one on each of R_{yp} and R_{zp} . Thus a correspondence is induced between

the generators of the two ruled surfaces in such a way that corresponding generators intersect. It is such pairs of ruled surfaces that we wish to consider.

It will be assumed that $P_y, P_z, P_\rho, P_\sigma$ are linearly independent points and that the curves generated by them are all immersed in the space S_3 . Moreover, $R_{y\rho}$ and $R_{z\rho}$ shall be taken to be non-developable ruled surfaces. Consequently, the determinant

$$\begin{vmatrix} y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ z^{(1)} & z^{(2)} & z^{(3)} & z^{(4)} \\ \rho^{(1)} & \rho^{(2)} & \rho^{(3)} & \rho^{(4)} \\ \sigma^{(1)} & \sigma^{(2)} & \sigma^{(3)} & \sigma^{(4)} \end{vmatrix}$$

is different from zero, and it is possible to determine the coefficients in the system of differential equations

$$\begin{aligned} (1) \quad y' &= a_{11}y + a_{12}z + a_{13}\rho + a_{14}\sigma, \\ z' &= a_{21}y + a_{22}z + a_{23}\rho + a_{24}\sigma, \\ \rho' &= a_{31}y + a_{32}z + a_{33}\rho + a_{34}\sigma, \\ \sigma' &= a_{41}y + a_{42}z + a_{43}\rho + a_{44}\sigma \end{aligned}$$

so that $(y^{(i)}, z^{(i)}, \rho^{(i)}, \sigma^{(i)})$ ($i = 1, 2, 3, 4$) are four sets of solutions. For example, let us substitute each of these sets in turn in the first equation of system (1). The resulting system of four equations is consistent, and thus the coefficients of the first of equations (1) can be determined. Similarly, we can determine the coefficients in each of the other three equations.

When the surfaces $R_{y\rho}$ and $R_{z\rho}$ are given, system (1) is not uniquely determined. New reference curves $C_{\bar{y}}, C_{\bar{z}}, C_{\bar{\rho}}, C_{\bar{\sigma}}$ may be introduced by the transformation

$$\begin{aligned}
 (2) \quad y &= a\bar{y} + b\bar{p}, \\
 z &= c\bar{z} + d\bar{p}, \\
 \rho &= e\bar{p}, \\
 \sigma &= f\bar{y} + g\bar{z} + h\bar{p} + k\bar{\sigma}
 \end{aligned}$$

for which the determinant

$$\begin{vmatrix}
 a & 0 & b & 0 \\
 0 & c & d & 0 \\
 0 & 0 & e & 0 \\
 f & g & h & k
 \end{vmatrix}$$

does not vanish. It follows immediately that none of the coefficients a, c, e, k is zero. We may also change the independent variable t by the transformation

$$(3) \quad u = U(t) \quad (U' \neq 0).$$

Under the transformations (2), (3) the surfaces $R_{y\rho}$ and $R_{z\rho}$ remain unchanged; but system (1) goes over into a similar system whose coefficients, indicated by dashes, are given by the following equations:

$$\begin{aligned}
 aU'\bar{a}_{11} &= a_{11}a + a_{14}f - a' - b\bar{a}_{31}, \\
 aU'\bar{a}_{12} &= a_{12}c + a_{14}g - b\bar{a}_{32}, \\
 aU'\bar{a}_{13} &= a_{11}b + a_{12}d + a_{13}e + a_{14}h - b' - b\bar{a}_{33}, \\
 aU'\bar{a}_{14} &= a_{14}k - b\bar{a}_{34}, \\
 cU'\bar{a}_{21} &= a_{21}a + a_{24}f - d\bar{a}_{31}, \\
 cU'\bar{a}_{22} &= a_{22}c + a_{24}g - c' - d\bar{a}_{32}, \\
 cU'\bar{a}_{23} &= a_{21}b + a_{22}d + a_{23}e + a_{24}h - d' - d\bar{a}_{33}, \\
 cU'\bar{a}_{24} &= a_{24}k - d\bar{a}_{34},
 \end{aligned}$$

$$eU'\bar{a}_{31} = a_{31}a + a_{34}f,$$

$$eU'\bar{a}_{32} = a_{32}c + a_{34}g,$$

$$eU'\bar{a}_{33} = a_{31}b + a_{32}d + a_{33}e + a_{34}h - o',$$

$$eU'\bar{a}_{34} = a_{34}k,$$

$$kU'\bar{a}_{41} = a_{41}a + a_{44}f - f' - f\bar{a}_{11} - g\bar{a}_{21} - h\bar{a}_{31},$$

$$kU'\bar{a}_{42} = a_{42}c + a_{44}g - g' - f\bar{a}_{12} - g\bar{a}_{22} - h\bar{a}_{32},$$

$$kU'\bar{a}_{43} = a_{41}b + a_{42}d + a_{43}e + a_{44}h - h' - f\bar{a}_{13} - g\bar{a}_{23} - h\bar{a}_{33},$$

$$kU'\bar{a}_{44} = a_{44}k - k' - f\bar{a}_{14} - g\bar{a}_{24} - h\bar{a}_{34}.$$

2. Non-special pairs of ruled surfaces. This section is devoted to the study of pairs of ruled surfaces $R_{y\rho}$, $R_{z\rho}$ with intersecting generators $\chi_{y\rho}$, $\chi_{z\rho}$ and such that the tangent to the curve of intersection C_ρ is not coplanar with the generators at each point. These surfaces shall be referred to as non-special pairs of ruled surfaces. By making a suitable choice of the reference curves, the geometrical characterization of which will be shown, it is possible to obtain a simple canonical form for system (1). The effect of the most general transformation (2), (3) on this canonical system is observed, and the most general transformation (2), (3) preserving its characterization is obtained.

The generators $\chi_{y\rho}$, $\chi_{z\rho}$ determine a plane $\pi_{yz\rho}$. As t varies we have a one-parameter family of planes, the envelope of which is a developable surface. The following theorem can now be proved:

A necessary and sufficient condition that the derivative point P_ρ' of P_ρ lie in the plane $\pi_{yz\rho}$ is that the characteristic line of the one-parameter family of planes pass through P_ρ .

For, the equation of the one-parameter family of planes is the fourth order determinant

$$(4) \quad (X, y, z, \rho) = 0,$$

where the vector X represents the coordinates of a variable point P_X in the plane $\pi_{yz\rho}$. The characteristic line of this family is defined by

$$(X, y, z, \rho) = 0, \quad (X, y, z, \rho') = 0,$$

where the prime indicates differentiation with respect to the variable t . Obviously the condition for this line to pass through P_ρ is that

$$(5) \quad (\rho, y, z, \rho') = 0,$$

that is to say, that $P_{\rho'}$ be coplanar with P_y, P_z, P_ρ . Conversely, if equation (5) holds, the characteristic line passes through P_ρ . Moreover, the ruled surfaces $R_{y\rho}$ and $R_{z\rho}$ are integral surfaces of system (1) and thus the following result is readily justified:

The characteristic line of the one-parameter family of planes (4) passes through P_ρ if and only if the coefficient a_{34} in system (1) is zero.

In the following considerations we shall assume that $P_{\rho'}$ does not lie in the plane $\pi_{yz\rho}$ and consequently the characteristic line of the family of planes (4) does not pass through P_ρ .

For t fixed, the plane

$$(X, y, z, \rho)' = 0$$

intersects the generator $\lambda'_{y\rho}$ in the point whose coordinates are

$a_{34}y - a_{14}p$. Thus we have the theorem:

A necessary and sufficient condition that the characteristic line of the family of planes (4) intersect the generators $\chi_{y\rho}$ and $\chi_{z\rho}$ in P_y and P_z respectively is that both the coefficients a_{14} , a_{24} in system (1) be identically zero.

Let the curves of reference C_y , C_z be taken so that the characteristic line intersects the generators $\chi_{y\rho}$ and $\chi_{z\rho}$ in the points P_y and P_z respectively. Moreover, let $P_{\rho'}$ be identical with P_{σ} . Then the differential equations (1) take the form

$$\begin{aligned} (6) \quad y' &= a_{11}y + a_{12}z + a_{13}p, \\ z' &= a_{21}y + a_{22}z + a_{23}p, \\ \rho' &= a_{34}\sigma, \\ \sigma' &= a_{41}y + a_{42}z + a_{43}p + a_{44}\sigma, \end{aligned}$$

where $a_{34} \neq 0$. In order to obtain a canonical form for the system of differential equations (6), we make the following transformations which preserve the geometrical location of P_y , P_z and P_{ρ} .

$$\begin{aligned} (7) \quad y &= a\bar{y}, \\ z &= c\bar{z}, \\ \rho &= e\bar{\rho}, \\ \sigma &= f\bar{y} + g\bar{z} + h\bar{\rho} + k\bar{\sigma}, \end{aligned}$$

$$(8) \quad u = U(t) \quad (U' \neq 0).$$

Under these transformations, system (6) goes over into a similar system whose coefficients, indicated by dashes, are given by

$$\begin{aligned} aU'\bar{a}_{11} &= a_{11}a - a', \\ aU'\bar{a}_{12} &= a_{12}c, \\ aU'\bar{a}_{13} &= a_{13}e, \\ aU'\bar{a}_{14} &= 0, \end{aligned}$$

$$\begin{aligned}
cU'\bar{a}_{21} &= a_{21}a, & eU'\bar{a}_{31} &= a_{34}f, \\
cU'\bar{a}_{22} &= a_{22}c - c', & eU'\bar{a}_{32} &= a_{34}g, \\
cU'\bar{a}_{23} &= a_{23}e, & eU'\bar{a}_{33} &= a_{34}h - e', \\
cU'\bar{a}_{24} &= 0, & eU'\bar{a}_{34} &= a_{34}k, \\
\\
kU'\bar{a}_{41} &= a_{41}a + a_{44}f - f' - f\bar{a}_{11} - g\bar{a}_{21} - h\bar{a}_{31}, \\
kU'\bar{a}_{42} &= a_{42}c + a_{44}g - g' - f\bar{a}_{12} - g\bar{a}_{22} - h\bar{a}_{32}, \\
kU'\bar{a}_{43} &= a_{43}e + a_{44}h - h' - f\bar{a}_{13} - g\bar{a}_{23} - h\bar{a}_{33}, \\
kU'\bar{a}_{44} &= a_{44}k - k' - h\bar{a}_{34}.
\end{aligned}$$

Since we wish $P_{\bar{\sigma}}$ to coincide with $P_{\bar{f}'}$, it follows that $\bar{a}_{31} = \bar{a}_{32} = \bar{a}_{33} = 0$. Hence $f = g = 0$, and $e' = a_{34}h$. It is possible to specialize the transformations (7), (8) to reduce (6) to a simple canonical form. If e and h are taken as a pair of non-zero solutions of the differential equations

$$e' = a_{34}h, \quad h' = a_{43}e + a_{44}h,$$

then $\bar{a}_{35} = \bar{a}_{43} = 0$. Similarly, $\bar{a}_{11} = \bar{a}_{22} = 0$ by taking a and c as non-zero solutions of the equations

$$a' = a_{11}a, \quad c' = a_{22}c.$$

Let k be taken as a non-zero solution of the equation

$$k' = a_{44}k - h,$$

where h has been previously determined. Moreover, let $u = U(t)$ be a solution of

$$eU' = a_{34}k,$$

where e and k have already been determined. Then it is easily seen that $\bar{a}_{44} = 0$ and $\bar{a}_{34} = 1$. When these reductions have been

made, system (6) reduces to the following simple form:

$$(9) \quad \begin{aligned} y' &= b_{12}z + b_{13}\rho, \\ z' &= b_{21}y + b_{23}\rho, \\ \rho' &= \sigma, \\ \sigma' &= b_{41}y + b_{42}z. \end{aligned}$$

Now let it be assumed that the original system of differential equations (1) is in the canonical form (9). We shall proceed to determine the most general transformation (7), (8) which preserves the analytic form of system (9). The effect of the transformations (7), (8) on (9) is to give a similar system in which the coefficients, indicated by dashes, are given by

$$\begin{aligned} aU'\bar{b}_{11} &= -a', & cU'\bar{b}_{21} &= b_{21}a, \\ aU'\bar{b}_{12} &= b_{12}c, & cU'\bar{b}_{22} &= -c', \\ aU'\bar{b}_{13} &= b_{13}e, & cU'\bar{b}_{23} &= b_{23}e, \\ aU'\bar{b}_{14} &= 0, & cU'\bar{b}_{24} &= 0, \\ eU'\bar{b}_{31} &= f, & kU'\bar{b}_{41} &= b_{41}a - f' - f\bar{b}_{11} - g\bar{b}_{21} - h\bar{b}_{31}, \\ eU'\bar{b}_{32} &= g, & kU'\bar{b}_{42} &= b_{42}c - g' - f\bar{b}_{12} - g\bar{b}_{22} - h\bar{b}_{32}, \\ eU'\bar{b}_{33} &= h - e', & kU'\bar{b}_{43} &= -h' - f\bar{b}_{13} - g\bar{b}_{23} - h\bar{b}_{33}, \\ eU'\bar{b}_{34} &= k, & kU'\bar{b}_{44} &= -k' - h\bar{b}_{34}. \end{aligned}$$

Since the coefficients b_{11} , b_{22} , b_{31} , b_{32} must be zero before and after the transformations, it follows that a and c are arbitrary constants different from zero, while f and g are identically zero. Also, $b_{33} = \bar{b}_{33} = 0$ requires that e be a non-zero solution of the equation $e' = h$. Demanding that $b_{43} = \bar{b}_{43} = 0$, it is seen that h is an arbitrary constant. Hence $e = ht + \alpha$, where α is a constant different from zero. Moreover, by taking k as a non-zero solution of $k' = -h$ and then taking $u = U(t)$ to

be a solution of $eU' = k$, it follows that $b_{44} = \bar{b}_{44} = 0$ and $b_{34} = \bar{b}_{34} = 1$ as desired. Thus it is evident that $k = -(ht + \beta)$ where β is a constant not zero and $u = U(t)$ is a solution of the equation

$$U' = \frac{ht + \beta}{ht + \alpha}.$$

Let us now consider system (9). The first equation states that the point y' lies on the generator $\lambda'_{z\rho}$. Evidently the coefficient b_{12} is different from zero, for otherwise $R_{y\rho}$ would be developable contrary to our original assumption. Similarly, the coefficient b_{21} is different from zero. It is easily seen that the tangent planes to $R_{y\rho}$ at P_y and P_ρ are the planes $\pi_{yz\rho}$ and $\pi_{y\rho\rho'}$, respectively. Likewise it is observed that the planes $\pi_{yz\rho}$ and $\pi_{z\rho\rho'}$ are tangent to $R_{z\rho}$ at P_z and P_ρ respectively. The planes $\pi_{z\rho\rho'}$, $\pi_{y\rho\rho'}$ obviously intersect in the tangent line to C_ρ at P_ρ .

The equation of the curved asymptotics on $R_{y\rho}$ can easily be found. Let P_φ , where $\varphi = \rho + \lambda y$, be an arbitrary point on the generator $\lambda'_{y\rho}$. Then differentiating twice and making use of system (9), we find

$$\varphi' = \lambda'y + b_{12}\lambda z + b_{13}\lambda\rho + \sigma,$$

$$\varphi'' = (\quad)y + (2b_{12}\lambda' + b_{12}'\lambda + b_{42})z + (\quad)\rho + b_{13}\lambda\sigma.$$

The curve C_φ will be an asymptotic curve on $R_{y\rho}$ if the osculating plane of C_φ at every point coincides with the tangent plane to $R_{y\rho}$ at that point. Demanding that the points ρ , φ , φ' , φ'' be coplanar, we obtain the relation

$$2b_{12}\lambda' = b_{12}b_{13}\lambda^2 - b_{12}'\lambda - b_{42}.$$

This Riccati equation then defines the curved asymptotics on $R_{y\rho}$. It is evident that C_ρ is an asymptotic curve on $R_{y\rho}$ in case $b_{42} = 0$; also since b_{12} is different from zero, C_y is an asymptotic curve in case $b_{13} = 0$. In a similar manner, letting $\Psi = \rho + \mu z$, we find the equation of the curved asymptotics on $R_{z\rho}$ to be

$$2b_{21}\mu' = b_{21}b_{23}\mu^2 - b_{21}'\mu - b_{41}.$$

The curves C_ρ , C_z are asymptotic curves on $R_{z\rho}$ in case $b_{41} = 0$, $b_{23} = 0$ respectively. However, it is important to note that C_y and C_z cannot be asymptotic curves simultaneously. For then it is obvious from (9) that C_y and C_z would be straight lines, contrary to our original assumption. Moreover, the curve C_ρ cannot be an asymptotic curve on both $R_{y\rho}$ and $R_{z\rho}$, since at P_ρ the tangent planes to $R_{y\rho}$ and $R_{z\rho}$ are distinct. Hence b_{41} and b_{42} are not zero simultaneously.

The characteristic line of the one-parameter family of planes defined by (4) joins the points P_y , P_z . Any point P_x on this line except P_y is given by $x = z + uy$, when u is a parameter independent of t . In order to determine the edge of regression of the developable surface enveloped by the plane of the corresponding generators, let $u = u(t)$ and demand that x' be a linear combination of y and z . Thus it follows that $b_{23} + ub_{13} = 0$. Consequently, the edge of regression of the developable surface is the curve C_x , where

$$x = z - \frac{b_{23}}{b_{13}} y.$$

It is easily seen that the osculating plane to C_ρ at P_ρ intersects the line χ_{yz} in the focal point P_x in case $b_{13}b_{41} + b_{23}b_{42} = 0$.

It is well known that every ruled surface is an integral ruled surface of a fundamental system of equations having the form

$$(10) \quad \begin{aligned} y'' + p_{11}y' + p_{12}p' + p_{13}y + p_{14}p &= 0, \\ p'' + p_{21}y' + p_{22}p' + p_{23}y + p_{24}p &= 0. \end{aligned}$$

In order to obtain system (10) from system (1), it is sufficient to differentiate the first two of equations (1) and eliminate z, z', σ, σ' . However, it is more convenient to start with the canonical system (9). We find that the coefficients of (10) for our ruled surface R_{yp} are given by the following formulas in terms of the coefficients of (9):

$$\begin{aligned} p_{11} &= -\frac{b_{12}'}{b_{12}}, & p_{12} &= -b_{13}, & p_{13} &= -b_{12}b_{21}', \\ p_{14} &= \frac{b_{12}'b_{13}}{b_{12}} - b_{12}b_{23} - b_{13}', \\ p_{21} &= -\frac{b_{42}}{b_{12}}, & p_{22} &= 0, & p_{23} &= -b_{41}, & p_{24} &= \frac{b_{13}b_{42}}{b_{12}}. \end{aligned}$$

The analogous system of differential equations for R_{zp} is obtained in a similar manner, and the coefficients are given by the following formulas:

$$\begin{aligned} q_{11} &= -\frac{b_{21}'}{b_{21}}, & q_{12} &= -b_{23}, & q_{13} &= -b_{12}b_{21}', \\ q_{14} &= \frac{b_{21}'b_{23}}{b_{21}} - b_{13}b_{21} - b_{23}', \\ q_{21} &= -\frac{b_{41}}{b_{21}}, & q_{22} &= 0, & q_{23} &= -b_{42}, & q_{24} &= \frac{b_{23}b_{41}}{b_{21}}. \end{aligned}$$

3. Special pairs of ruled surfaces. This section is devoted to the study of pairs of ruled surfaces R_{yp}, R_{zp} with intersecting generators and such that the tangent to the curve of

intersection C_P lies in the plane of the generators at each point. These surfaces shall be referred to as special pairs of ruled surfaces. By choosing the fundamental curves of reference on the two surfaces suitably, it is possible to reduce system (1) to a relatively simple canonical form. The effect of the transformations (2) and (3) on this canonical system is observed, and the most general transformation preserving it is obtained.

The point P_P lies in the plane π_{yzP} . Consequently the coefficient a_{34} in system (1) must be zero. It is well known that every plane through a generator of a non-developable ruled surface is tangent to the surface at just one point of the generator. Hence it is evident that the tangents to the curves C_Y , C_Z do not lie in the plane π_{yzP} . Thus for any particular choice of the reference curve C_Y , the curve C_Z can be chosen so that the tangents to C_Y and C_Z at P_Y and P_Z respectively intersect in a point which shall be taken to be P_σ . Then the differential equations (1) have the form

$$\begin{aligned}
 (11) \quad y' &= a_{14}\sigma, \\
 z' &= a_{24}\sigma, \\
 p' &= a_{31}y + a_{32}z + a_{33}p, \\
 \sigma' &= a_{41}y + a_{42}z + a_{43}p + a_{44}\sigma.
 \end{aligned}$$

It is immediately evident that none of the coefficients a_{14} , a_{24} , a_{31} , a_{32} are zero. Applying the transformations (2), (3) to (11), we obtain a similar system whose coefficients, indicated by dashes, are given by

$$\begin{aligned}
 aU'\bar{a}_{11} &= a_{14}f - a' - b\bar{a}_{31}, & cU'\bar{a}_{21} &= a_{24}f - d\bar{a}_{31}, \\
 aU'\bar{a}_{12} &= a_{14}g - b\bar{a}_{32}, & cU'\bar{a}_{22} &= a_{24}g - c' - d\bar{a}_{32},
 \end{aligned}$$

$$\begin{aligned}
aU'\bar{a}_{13} &= a_{14}h - b' - b\bar{a}_{33}, & cU'\bar{a}_{23} &= a_{24}h - d' - d\bar{a}_{33}, \\
aU'\bar{a}_{14} &= a_{14}k, & cU'\bar{a}_{24} &= a_{24}k, \\
eU'\bar{a}_{31} &= a_{31}a, & eU'\bar{a}_{33} &= a_{31}b + a_{32}d + a_{33}e - e', \\
eU'\bar{a}_{32} &= a_{32}c, & eU'\bar{a}_{34} &= 0, \\
kU'\bar{a}_{41} &= a_{41}a + a_{44}f - f' - f\bar{a}_{11} - g\bar{a}_{21} - h\bar{a}_{31}, \\
kU'\bar{a}_{42} &= a_{42}c + a_{44}g - g' - f\bar{a}_{12} - g\bar{a}_{22} - h\bar{a}_{32}, \\
kU'\bar{a}_{43} &= a_{41}b + a_{42}d + a_{43}e + a_{44}h - h' - f\bar{a}_{13} - g\bar{a}_{23} - h\bar{a}_{33}, \\
kU'\bar{a}_{44} &= a_{44}k - k' - f\bar{a}_{14} - g\bar{a}_{24}.
\end{aligned}$$

Demanding that $P_{\bar{y}}$ be identical with $P_{\bar{\sigma}}$ and $P_{\bar{z}}$ be identical with $P_{\bar{\sigma}}$, it is immediately seen that the coefficients of the transformation (2) must satisfy the following relations:

$$\begin{aligned}
a_{14}f - a' - b\bar{a}_{31} &= 0, & a_{24}f - d\bar{a}_{31} &= 0, \\
a_{14}g - b\bar{a}_{32} &= 0, & a_{24}g - c' - d\bar{a}_{32} &= 0, \\
a_{14}h - b' - b\bar{a}_{33} &= 0, & a_{24}h - d' - d\bar{a}_{33} &= 0.
\end{aligned}$$

In order to reduce system (11) to a simple canonical form, let us suppose that none of the curves of reference $C_{\bar{y}}$, $C_{\bar{z}}$, $C_{\bar{\sigma}}$ is a straight line. Now let $u = U(t)$ be an arbitrary function of t , and choose b, d, e, h as a set of non-zero solutions of the differential equations

$$\begin{aligned}
b' &= a_{14}h, \\
d' &= a_{24}h, \\
e' &= a_{31}b + a_{32}d + a_{33}e, \\
h' &= a_{41}b + a_{42}d + a_{43}e + a_{44}h.
\end{aligned}$$

It follows that $\bar{a}_{13} = \bar{a}_{23} = \bar{a}_{33} = \bar{a}_{43} = 0$. Moreover, let a and c be non-zero solutions of the equations

$$a_{24}a' = \bar{a}_{31}(a_{14}d - a_{24}b), \quad a_{14}c' = \bar{a}_{32}(a_{24}b - a_{14}d),$$

where $b, d, u = U(t)$ have been previously determined. Finally, it is possible to make $\bar{a}_{44} = 0$ by choosing k as a non-zero solution of the equation

$$k' = a_{44}k - f\bar{a}_{14} - g\bar{a}_{24}.$$

With these choices of the coefficients in the transformation (2), system (11) reduces to the following simple canonical form:

$$(12) \quad \begin{aligned} y' &= b_{14}\sigma, \\ z' &= b_{24}\sigma, \\ \rho' &= b_{31}y + b_{32}z, \\ \sigma' &= b_{41}y + b_{42}z. \end{aligned}$$

From a consideration of system (12), it is observed that the curves C_y , C_z , and C_σ all lie in the same plane. Moreover, by our assumption, none of these curves is a straight line. Consequently, the coefficients b_{41} , b_{42} are different from zero, and $b_{41} \neq mb_{42}$, where m is a constant different from zero. In particular, b_{42} , b_{41} are not constants simultaneously.

Let us not suppose that our original system of differential equations (11) is in the canonical form (12). It is important to determine the most general transformation (2), (3) which preserves the analytic form of system (12). The effect of the transformations (2), (3) on (12) is to give a system similar to (1) in which the coefficients, indicated by dashes, are given by

$$\begin{aligned} aU'\bar{b}_{11} &= b_{14}f - a' - b\bar{b}_{31}, \\ aU'\bar{b}_{12} &= b_{14}g - b\bar{b}_{32}, \\ aU'\bar{b}_{13} &= b_{14}h - b' - b\bar{b}_{33}, \\ aU'\bar{b}_{14} &= b_{14}k, \end{aligned}$$

$$\begin{aligned}
eU'\bar{b}_{31} &= b_{31}a, & cU'\bar{b}_{21} &= b_{24}f - d\bar{b}_{31}, \\
eU'\bar{b}_{32} &= b_{32}c, & cU'\bar{b}_{22} &= b_{24}g - c' - d\bar{b}_{32}, \\
eU'\bar{b}_{33} &= b_{31}b + b_{32}d - e', & cU'\bar{b}_{23} &= b_{24}g - d' - d\bar{b}_{33}, \\
eU'\bar{b}_{34} &= 0, & cU'\bar{b}_{24} &= b_{24}k, \\
kU'\bar{b}_{41} &= b_{41}a - f' - f\bar{b}_{11} - g\bar{b}_{21} - h\bar{b}_{31}, \\
kU'\bar{b}_{42} &= b_{42}c - g' - f\bar{b}_{12} - g\bar{b}_{22} - h\bar{b}_{32}, \\
kU'\bar{b}_{43} &= b_{41}b + b_{42}d - h' - f\bar{b}_{13} - g\bar{b}_{23} - h\bar{b}_{33}, \\
kU'\bar{b}_{44} &= -k' - f\bar{b}_{14} - g\bar{b}_{24}.
\end{aligned}$$

The most general transformation preserving (12) is obtained by taking $u = U(t)$ to be an arbitrary function of t and letting b, d, e, h be any set of non-zero solutions of

$$\begin{aligned}
b' &= b_{14}h, \\
d' &= b_{24}h, \\
e' &= b_{31}b + b_{32}d, \\
h' &= b_{41}b + b_{42}d.
\end{aligned}$$

Moreover, a and c are given by the differential equations

$$b_{24}a' = \bar{b}_{31}(b_{14}d - b_{24}d), \quad b_{14}c' = \bar{b}_{32}(b_{24}b - b_{14}d).$$

The coefficients f and g must satisfy the relations

$$b_{24}f - d\bar{b}_{31} = 0, \quad b_{14}g - b\bar{b}_{32} = 0,$$

and k is to be taken as any non-zero solution of the equation

$$k' = -f\bar{b}_{14} - g\bar{b}_{24}.$$

The equation of the curved asymptotics on $R_{y\rho}$ is easily found to be

$$(13) \quad 2b_{14}b_{32}u' = -b_{32}(b_{14}b_{31} + b_{24}b_{32}) + (b_{14}b_{32}' - b_{14}'b_{32})u + b_{14}''b_{42}u^2.$$

The curve C_p is an asymptotic curve in case $b_{14}b_{31} + b_{24}b_{32} = 0$. Since the coefficient b_{42} is different from zero, the curve C_y is not an asymptotic curve. The corresponding equation for the surface R_{zp} is obtained from (13) by a substitution on the subscripts of the b 's in which 1 goes into 2 and 2 into 1, while 3 and 4 remain unchanged. Since b_{41} is different from zero, C_z cannot be an asymptotic curve. The condition for C_p to be asymptotic is the same as for the case of R_{yp} . Thus it follows that if C_p is an asymptotic curve on one ruled surface it is also an asymptotic curve on the other.

Consider now the family of planes (4). For special pairs of ruled surfaces, the characteristic line of the family joins the points P_p and $P_z + ky$, where $b_{14}k + b_{24} = 0$. Any point P_x on the line joining these two points is defined by

$$x = p + u(z + ky),$$

where u is a parameter independent of t . In order to find the edge of regression of the family of planes (4), let $u = u(t)$ and demand that x' be a linear combination of p and $(z + ky)$. It follows that the edge of regression of the developable in S_3 enveloped by the plane of the corresponding generators is the curve C_x , where $b_{14}k'u = b_{14}'b_{31} + b_{24}b_{32}$. If C_p is an asymptotic curve, the plane of the two intersecting generators osculates the curve of intersection.

The differential equations analogous to (10) for R_{yp} can be easily obtained from (12). The coefficients corresponding to those in (10) are given by the following formulas in terms of the coefficients of (12):

$$\begin{aligned}
 (14) \quad p_{11} &= -\frac{b_{14}'}{b_{14}}, & p_{12} &= -\frac{b_{14}b_{42}}{b_{32}}, \\
 p_{13} &= \frac{b_{14}(b_{31}b_{42} - b_{32}b_{41})}{b_{32}}, & p_{14} &= 0, \\
 p_{21} &= -\frac{(b_{14}b_{31} + b_{24}b_{32})}{b_{14}}, & p_{22} &= -\frac{b_{32}'}{b_{32}}, \\
 p_{23} &= \frac{b_{31}b_{32}' - b_{31}'b_{32}}{b_{32}}, & p_{24} &= 0.
 \end{aligned}$$

The corresponding coefficients of the differential equations analogous to (10) for R_{zp} are obtained from (14) by a substitution on the subscripts of the b 's in which 1 and 2 are interchanged while 3 and 4 remain fixed.

4. Ruled surfaces associated with a non-ruled surface in ordinary space. This section is devoted to a consideration of pairs of ruled surfaces with intersecting generators which are associated with a proper non-ruled surface S immersed in space S_3 . By taking a one-parameter family of lines of a congruence Γ_1 , an interesting example of a non-special pair of ruled surfaces is obtained. The defining system of differential equations is then determined. Next it is shown that the asymptotic tangents of S at points along a curve on the surface constitute a special pair of ruled surfaces. The differential equations of this configuration are easily obtained.

Let the four homogeneous coordinates $x^{(1)}, \dots, x^{(4)}$ of an arbitrary point P_x on a surface S be given as single-valued analytic functions of two independent variables u and v . Let us assume that S is a proper non-ruled surface immersed in S_3 and is referred to its asymptotic net. Then the four functions x are a fundamental system of solutions of a completely integrable system of partial differential equations which may be taken in the form

$$(15) \quad \begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v \end{aligned} \quad (\theta = \log \beta \gamma).$$

The coefficients of (15) satisfy three integrability conditions:

$$\begin{aligned} \theta_{uvv} &= (\gamma \varphi)_u + 2q_u + \theta_v \theta_{uv} - \beta \gamma \gamma, \\ \theta_{uuv} &= (\beta \psi)_v + 2p_v + \theta_u \theta_{uv} - \beta \gamma \varphi, \\ p_{vv} - \theta_v p_v + \beta q_v + 2q \beta_v &= q_{uu} - \theta_u q_u + \gamma p_u + 2p \gamma_u, \end{aligned}$$

where

$$\varphi = (\log \beta \gamma^2)_u, \quad \psi = (\log \beta^2 \gamma)_v.$$

Consider now an integral surface S of equations (15) with the parametric vector equation $x = x(u, v)$. Let P_x be a point on S and consider any line χ_1 through P_x but not lying in the tangent plane of S at P_x . Such a line χ_1 may be regarded as determined by the point P_x and a point P_y defined by

$$(16) \quad y = -ax_u - bx_v + x_{uv},$$

where the coefficients a, b are scalar functions of u, v . As u, v vary, the line χ_1 generates a congruence Γ_1 . It is a well-known theorem that the lines of a congruence Γ_1 can be assembled into a one-parameter family of developable surfaces in two ways (ordinarily determinate and distinct) so that there is one developable of each family through each line χ_1 . The curves in which the developables of the congruence Γ_1 intersect S are called Γ_1 -curves.

An arbitrary curve C_p on S (except the curves $u = \text{const-}$ ant) may be defined by expressing v as a function of u in the form $v = v(u)$. For convenience in notation, let P_p denote P_x along C . If C_p is not a Γ_1 -curve or an asymptotic curve on S , then du, dv

do not satisfy

$$(F - 2a\beta + \beta\gamma)du^2 - (b_v - a_u)du dv - (G - 2b\gamma + \gamma\varphi)dv^2 = 0,$$

where

$$F = p - b_u + b\theta_u - b^2 + a\beta, \quad G = q - a_v + a\theta_v - a^2 + b\gamma.$$

Moreover, the ruled surface $R_{y\rho}$ of the congruence Γ_1 that intersects the surface S in the curve C_ρ is not developable.

At each point of the curve C_ρ let us consider the asymptotic u-tangent determined by the points P_ρ and P_z , where P_z is identical with P_{x_u} . The locus of these tangents is a non-developable ruled surface. It is easily observed that the ruled surface of asymptotic u-tangents at points of C_ρ and the ruled surface of lines of the congruence Γ_1 along C_ρ are a pair of ruled surfaces with intersecting generators. Moreover, this system of surfaces is non-special.

To determine the differential equations analogous to (1) for this system, let P_σ coincide with P_{x_u} . We calculate the total derivatives of y, z, ρ, σ with respect to u , using the formula $y' = y_u + \lambda y_v$, where λ is obtained by differentiating the equation $v = v(u)$. After some computation, we obtain a system of differential equations of the form (1), whose coefficients have the following values:

$$\begin{aligned} a_{11} &= -a\lambda - b + \theta_u + \theta_v\lambda, & a_{21} &= \lambda, \\ a_{12} &= C_2 - \frac{C_3}{\lambda} + (a - \frac{b}{\lambda})a_{11}, & a_{22} &= \theta_u - \frac{\beta}{\lambda} + \lambda(a - \frac{b}{\lambda}), \\ a_{13} &= C_1, & a_{23} &= p, \\ (17) \quad a_{14} &= \frac{C_3}{\lambda} + \frac{b}{\lambda} a_{11}, & a_{24} &= \frac{\beta}{\lambda} + b, \\ a_{31} &= 0, \quad a_{32} = 0, \quad a_{33} = 0, \quad a_{34} = 1, \\ a_{41} &= 2\lambda, \quad a_{42} = \theta_u + \gamma\lambda^2 - \frac{C_4}{\lambda} + 2\lambda(a - \frac{b}{\lambda}), \\ a_{43} &= p + q\lambda^2, & a_{44} &= \frac{C_4}{\lambda} + 2b, \end{aligned}$$

where

$$\begin{aligned}
 (18) \quad C_1 &= -ap - bq\lambda + p_v + \beta q + (q_u + \gamma p)\lambda, \\
 C_2 &= -a' - a\theta_u - b\gamma\lambda + \beta\gamma + \theta_{uv} + \chi\lambda, \\
 C_3 &= -a\beta - b' - \theta_v b\lambda + \pi + (\beta\gamma + \theta_{uv})\lambda, \\
 C_4 &= \beta + \theta_v\lambda^2 + \lambda', \\
 \pi &= p + \beta\gamma, \quad \chi = q + \gamma\varphi.
 \end{aligned}$$

Let us consider now the ruled surface $R_{z\rho}$ of asymptotic v -tangents at points of the curve C_ρ , and also the ruled surface $R_{y\rho}$ of the congruence Γ_1 that intersects S in C_ρ . These two surfaces have their corresponding generators intersecting and obviously form a non-special pair of ruled surfaces. Let the point P_z be identical with P_{xv} , and again let it be supposed that C_ρ is not an asymptotic u -curve or a Γ_1 -curve. Proceeding as before, we obtain a system of differential equations of the form (1) for this pair of surfaces. The coefficients of the system have the following values:

$$\begin{aligned}
 (19) \quad a_{11} &= -a\lambda - b + \theta_u + \theta_v\lambda, \quad a_{21} = 1, \\
 a_{12} &= -\lambda C_2 + C_3 + (b - a\lambda)a_{11}, \quad a_{22} = b - a\lambda - \gamma\lambda^2 + \theta_v\lambda, \\
 a_{13} &= C_1, \quad a_{23} = q\lambda, \\
 a_{14} &= C_2 + aa_{11}, \quad a_{24} = a + \gamma\lambda, \\
 a_{31} &= 0, \quad a_{32} = 0, \quad a_{33} = 0, \quad a_{34} = 1, \\
 a_{41} &= 2\lambda, \quad a_{42} = -\lambda(\theta_u + \gamma\lambda^2) + C_4 + 2b\lambda - 2a\lambda^2, \\
 a_{43} &= 1, \quad a_{44} = \theta_u + \gamma\lambda^2 + 2a\lambda,
 \end{aligned}$$

where C_1 , C_2 , C_3 and C_4 are as in (18).

It is evident that corresponding to each pair of values of a and b in (16), there are defined two non-special pairs of ruled surfaces. Therefore as a and b are allowed to vary independently, there are two two-parameter families of non-special

pairs of ruled surfaces. Among these it is important to note in particular those defined for canonical lines λ_1 . If $a = \sqrt{2}$ and $b = \varphi/2$, the corresponding line λ_1 is called a directrix of Wilczynski. Then the generators of $R_{y\rho}$ constitute a one-parameter family of these directrices. If $a = \sqrt{3}$ and $b = \varphi/3$, the corresponding congruence Γ_1 is called the axis congruence. In this case R_y is generated by a one-parameter family of axes. The pairs of ruled surfaces defined for the projective normal congruence are obtained by letting $a = 0$ and $b = 0$. In this case, the coefficients in (17) and (19) reduce to their simplest form.

A very simple but interesting example of a special pair of ruled surfaces may be obtained as follows. Consider a proper non-ruled surface S in S_3 defined by (15). Let C_ρ be any non-asymptotic curve on S whose equation is $v = v(u)$, and let P_ρ denote P_x on C_ρ . At any point on C_ρ , consider the u -tangent to S . As P_ρ goes along C_ρ , the u -tangent generates a non-developable ruled surface. Similarly, the locus of the v -tangents at points of the curve C_ρ is a non-developable ruled surface. This system of two ruled surfaces defined by the parametric tangents along C_ρ constitutes a special pair of ruled surfaces.

The differential equations for this system are easily obtained. For convenience, let $P_y, P_z, P_\rho, P_\sigma$ be identical with $P_{x_u}, P_{x_v}, P_x, P_{x_{uv}}$ respectively. Obviously these four points are linearly independent. We calculate the total derivatives of y, z, ρ, σ with respect to u , using the formula $y' = y_u + \lambda y_v$, where $dv/du = \lambda(u)$. Thus we arrive at a system of equations analogous to (1) whose coefficients have the following simple values:

$$\begin{aligned}
 a_{11} &= e_u, & a_{21} &= \gamma \lambda, & a_{31} &= 1, & a_{41} &= e_{uv} + \beta \gamma + \chi \lambda, \\
 a_{12} &= \beta, & a_{22} &= e_v \lambda, & a_{32} &= \lambda, & a_{42} &= \pi + \lambda(\beta \gamma + e_{uv}), \\
 (20) \quad a_{13} &= p, & a_{23} &= q \lambda, & a_{33} &= 0, & a_{43} &= p_v + \beta q + \lambda(q_u + \gamma p), \\
 a_{14} &= \lambda, & a_{24} &= 1, & a_{34} &= 0, & a_{44} &= e_u + e_v \lambda.
 \end{aligned}$$

The special pairs of ruled surfaces defined by taking the parametric tangents at points of the curve $C_{-\lambda}$, which is conjugate to C_λ , may be considered in the same manner. The coefficients for the corresponding system of differential equations are given by (20), where now λ is replaced by $-\lambda$.

5. The Klein-Segre transformation. The correspondence of lines in ordinary space into points on a hyperquadric Q_4 in a linear space S_5 provides us with some interesting relations for pairs of ruled surfaces with intersecting generators. Some of the essential features of this correspondence are summarized in the beginning of this section. To a pair of ruled surfaces in S_3 corresponds on Q_4 two curves $C_{\overline{y}}$, $C_{\overline{z}}$. If corresponding generators are not coplanar, it is possible to define a ruled surface $R_{\overline{yz}}$ in S_5 , which intersects Q_4 in the curves $C_{\overline{y}}$, $C_{\overline{z}}$. For pairs of ruled surfaces with intersecting generators, the surface actually lies on Q_4 . It is observed that the surface $R_{\overline{yz}}$ is developable if and only if the corresponding pair of ruled surfaces in S_3 is special. For non-special pairs of ruled surfaces, it is seen that certain two one-parameter families of flat pencils, which are associated with this system of ruled surfaces, can be employed to define two ruled surfaces with intersecting generators on Q_4 . Finally, it is noted that special pairs of ruled surfaces may be considered as

limiting cases of non-special pairs of ruled surfaces.

It is convenient to denote by R_3 the ordinary projective space with the straight line as generating element. If P_x and P_y are any two distinct points in S_3 , the pluckerian coordinates of the line joining them are given by

$$(21) \quad \omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{42}, \omega_{34},$$

where $\omega_{ij} = x_i y_j - x_j y_i$; ($i = 1, \dots, 4$). We define the symbol $(a\omega)$ by the formula

$$(a\omega) = a_{34}\omega_{12} + a_{42}\omega_{13} + a_{23}\omega_{14} + a_{14}\omega_{23} + a_{13}\omega_{42} + a_{12}\omega_{34} = 0.$$

The coordinates (21) satisfy the quadratic relation $(\omega\omega) = 0$.

The correspondence with which we shall be dealing is defined as follows. Let the six homogeneous coordinates ω of a line in ordinary ruled space R_3 be interpreted as the projective homogeneous coordinates of a point in a linear space S_5 , then to the lines of space R_3 correspond the points on the hyperquadric Q_4 whose equation is $(\omega\omega) = 0$ in the space S_5 , corresponding point and line having the same or proportional coordinates.

We note the following well-known theorems:

To a flat pencil of lines in ordinary ruled space R_3 corresponds a rectilinear generator of the hyperquadric Q_4 , each line of the pencil corresponding to a point of the generator.

To the ruled planes of space R_3 correspond the planar generators of one family of the hyperquadric Q_4 , and to the bundles of lines of R_3 correspond the planar generators of the other family of Q_4 .

It is easily observed that the correspondent of a ruled surface in space R_3 is a curve C on the hyperquadric Q_4 in space S_5 .

If the ruled surface is developable, then any two consecutive generators intersect determining a pencil of lines. Thus the tangents of the curve C on the hyperquadric Q_4 are generators of Q_4 .

Let us consider now pairs of ruled surfaces $R_{y\rho}$, $R_{z\sigma}$ with generators in one-to-one correspondence and corresponding generators not coplanar. Corresponding to a pair of generators $\lambda_{y\rho}$, $\lambda_{z\sigma}$ on $R_{y\rho}$, $R_{z\sigma}$ respectively, there are two distinct points $P_{\bar{y}}$, $P_{\bar{z}}$ on the hyperquadric Q_4 . Let corresponding points $P_{\bar{y}}$, $P_{\bar{z}}$ be joined by a straight line $\lambda_{\bar{y}\bar{z}}$. Then as t varies, the points $P_{\bar{y}}$, $P_{\bar{z}}$ generate curves $C_{\bar{y}}$, $C_{\bar{z}}$ respectively, and the locus of $\lambda_{\bar{y}\bar{z}}$ is a ruled surface $R_{\bar{y}\bar{z}}$. It is evident that $R_{\bar{y}\bar{z}}$ intersects Q_4 in the curves $C_{\bar{y}}$, $C_{\bar{z}}$.

Now let us consider non-special pairs of ruled surfaces. Using the previous notations, it is seen that corresponding to a pair of intersecting generators $\lambda_{y\rho}$, $\lambda_{z\rho}$ on $R_{y\rho}$, $R_{z\rho}$ respectively, there are two distinct points $P_{\bar{y}}$, $P_{\bar{z}}$ on the hyperquadric Q_4 . Moreover, to the pencil of lines determined by these generators corresponds on Q_4 a straight line $\lambda_{\bar{y}\bar{z}}$ joining $P_{\bar{y}}$, $P_{\bar{z}}$. As t varies, the points $P_{\bar{y}}$, $P_{\bar{z}}$ generate curves $C_{\bar{y}}$, $C_{\bar{z}}$, and the locus of $\lambda_{\bar{y}\bar{z}}$ is a ruled surface $R_{\bar{y}\bar{z}}$, which lies on Q_4 . It is evident that $R_{\bar{y}\bar{z}}$ is not developable. For, corresponding to two consecutive pencils of lines determined by two consecutive pairs of intersecting generators, there are on Q_4 two consecutive generators of $R_{\bar{y}\bar{z}}$. Since no two consecutive pencils of lines have a line in common, no two consecutive generators of $R_{\bar{y}\bar{z}}$ intersect. Thus $R_{\bar{y}\bar{z}}$ is a non-developable ruled surface.

For the case of special pairs of ruled surfaces, there is a corresponding ruled surface $R_{\bar{y}\bar{z}}$ on the hyperquadric Q_4 . A pair

of intersecting generators λ_{yp} , λ_{zp} determines a pencil of lines. Two consecutive pencils have a line in common, namely, the characteristic line of the family of planes (4). Thus two consecutive generators of $R_{\bar{y}\bar{z}}$ intersect in the point, which is the correspondent of the characteristic line of (4), and it follows immediately that $R_{\bar{y}\bar{z}}$ is developable. If the family of planes (4) osculates the curve of intersection of R_{yp} , R_{zp} , then to the developable surface of tangents of C_p corresponds on Q_4 the edge of regression of the developable surface $R_{\bar{y}\bar{z}}$.

Let us consider again non-special pairs of ruled surfaces. To the developable surface of tangents to the curve C_p corresponds a curve $C_{\bar{p}}$ on Q_4 . To the one-parameter family of flat pencils determined by the tangents of C_p and the generators λ_{yp} corresponds a ruled surface $R_{\bar{y}\bar{p}}$ on Q_4 . Similarly, we obtain on Q_4 another ruled surface $R_{\bar{z}\bar{p}}$ on Q_4 , which corresponds to the one-parameter family of flat pencils determined by the tangents of C_p and the generators λ_{zp} . Hence there are on Q_4 two ruled surfaces with intersecting generators. Two consecutive generators of the developable surface of tangents to C_p determine a pencil of lines. This pencil corresponds to a tangent line of the curve $C_{\bar{p}}$ on Q_4 . However, to a bundle of lines in S_3 corresponds a planar generator on Q_4 . Since the lines λ_{yp} , λ_{zp} , λ_{pp} , determine a bundle of lines in R_3 , it follows that the tangent to $C_{\bar{p}}$ at $P_{\bar{p}}$ lies in the plane determined by the lines $\lambda_{\bar{y}\bar{p}}$, $\lambda_{\bar{z}\bar{p}}$. Thus the system of a pair of ruled surfaces on Q_4 with intersecting generators is special.

Now assume t fixed, and let us consider the ruled plane determined by the generators λ_{yp} , λ_{zp} . This ruled plane corresponds to a planar generator on Q_4 . Then as t varies, the locus

is a one-parameter family of planar generators corresponding to the family of planes whose equation is $(X, y, z, p) = 0$. Moreover, since two consecutive planes of this family intersect in the characteristic line, the two corresponding planar generators on Q_4 intersect in the point which is the correspondant of the characteristic line. This point generates a curve which is the edge of regression of the one-parameter family of planar generators corresponding to the family of ruled planes in R_3 .

It was observed that for a non-special pair of ruled surfaces it was possible to define two ruled surfaces $R_{\bar{y}\bar{p}}$, $R_{\bar{z}\bar{p}}$ with intersecting generators on the hyperquadric Q_4 . Let us think for the moment of the intersecting generators as approaching coincidence. Then in the limit, the system of two ruled surfaces on Q_4 would reduce to a single ruled surface $R_{\bar{y}\bar{z}}$. Thus a special pair of ruled surfaces may be considered as a limiting case of a non-special pair of ruled surfaces.

VITA

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